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Electroweak vacuum geometry

Nathan F. Lepora

Blackett Laboratory, Imperial College
Prince Consort Road, London, SW7 2BZ, England, and
King's College, Cambridge University
Cambridge, CB2 1ST, England
E-mail: `N.F.Lepora@ic.ac.uk`

Tom W.B. Kibble

King's College, Cambridge University
Cambridge, CB2 1ST, England
E-mail: `T.Kibble@ic.ac.uk`

Abstract: We analyse symmetry breaking in the Weinberg-Salam model paying particular attention to the underlying geometry of the theory. In this context we find two natural metrics upon the vacuum manifold: an isotropic metric associated with the scalar sector, and a *squashed* metric associated with the gauge sector. Physically, the interplay between these metrics gives rise to many of the non-perturbative features of Weinberg-Salam theory.

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Contents

1. Introduction

Numerous experiments have shown that perturbative Electroweak interactions are described by the Weinberg-Salam theory of a broken isospin-hypercharge gauge symmetry. Within that theory the vacuum is instrumental in breaking the isospin-hypercharge $\mathrm{SU}(2)_I \times \mathrm{U}(1)_Y$ symmetry to a residual electromagnetic $\mathrm{U}(1)_Q$ theory. Physically, the symmetry is broken by the coupling between this vacuum and isospin-hypercharge gauge fields. This coupling induces mass for the W and Z components of the gauge fields, whilst the photon does not couple, remains massless and represents the residual gauge theory.

However, in describing the above symmetry breaking Weinberg-Salam theory also makes definite predictions about the vacuum structure. Owing to the $\mathrm{SU}(2)_I \times \mathrm{U}(1)_Y$ gauge symmetry, vacua are predicted to be degenerate and collectively form the vacuum *manifold*. This structure then implies the existence of non-perturbative solutions, for example the electroweak strings and the sphaleron. Thus in describing

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electromagnetism in conjunction with a W and Z sector, Weinberg-Salam theory *implies* the existence of further non-perturbative solutions related to the vacuum structure.

By explicit calculation the relevant vacuum manifold M is predicted to be a three-sphere. This three-sphere is related to the gauge structure by the familiar relation

$$M = S^3 \cong \frac{\mathrm{SU}(2)_I \times \mathrm{U}(1)_Y}{\mathrm{U}(1)_Q}\,. \tag{1.1}$$

Now a reasonable question is: what is special about this relation? For instance how does $\mathrm{SU}(2)_I \times \mathrm{U}(1)_Y/\mathrm{U}(1)_Q$ differ from $\mathrm{SO}(4)/\mathrm{SO}(3)$, which is also isomorphic to a three sphere.

Mathematically, the answer to this question is: $\mathrm{SO}(4)/\mathrm{SO}(3)$ and $\mathrm{SU}(2)_I \times \mathrm{U}(1)_Y/\mathrm{U}(1)_Q$ describe different *metrical structures* on the three-sphere. Both describe homogenous metrics, but they differ in the symmetry properties of the metric. Essentially $\mathrm{SO}(4)/\mathrm{SO}(3)$ is associated with a homogenous and isotropic metric on the three-sphere, whilst the other gives a homogenous and *anisotropic* metric.

Within this paper we examine how this mathematical structure relates to the vacuum structure. It turns out that the anisotropic $\mathrm{SU}(2)_I \times \mathrm{U}(1)_Y/\mathrm{U}(1)_Q$ metric is naturally induced by the gauge sector, with the degree of anisotropy specified by the gauge coupling constants. In addition the isotropic $\mathrm{SO}(4)/\mathrm{SO}(3)$ metric is naturally induced by the scalar sector of Weinberg-Salam theory. Thus eq. (1.1) is interpreted as appertaining to the gauge sector geometry of Weinberg-Salam theory, whilst $S^3 \cong \mathrm{SO}(4)/\mathrm{SO}(3)$ appertains to the scalar sector.

We then use this framework to interpret the non-perturbative solution spectrum in terms of the vacuum geometry. Such solutions are usually specified in terms of their boundary conditions on the vacuum manifold. Electroweak strings define embedded circles, whilst the sphaleron defines an embedded two-sphere. With respect to the geometry we show that these boundary conditions relate to totally geodesic submanifolds of the three-sphere with respect to *both* the scalar and gauge metrics. We also show that the scattering of electroweak strings relate to the holonomy of their boundary conditions with respect to these metrics.

2. Electroweak symmetry breaking

We start by quickly running through the symmetry breaking mechanism in electroweak theory. We use this discussion to make explicit some of the mathematical structure required for this paper.

The usual approach is taken, whereby a Lagrangian describes the interaction of the gauge fields $W^\mu \in \mathrm{su}(2)$ and $Y^\mu \in \mathrm{u}(1)$ with a scalar field $\Phi \in \mathbb{C}^2$. Minimisation of the scalar potential yields a vacuum with less gauge invariance than the original

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symmetry. Then orthogonal rotation of the W^μ and Y^μ fields gives the W, Z and photon basis of mass eigenstates with respect to this vacuum.

The scalar-gauge sector of Weinberg-Salam theory is described by the lagrangean

$$\mathcal{L} = -\frac{1}{4}\langle W^{\mu\nu} + Y^{\mu\nu}, W_{\mu\nu} + Y_{\mu\nu}\rangle + (\mathcal{D}^\mu\Phi)^\dagger(\mathcal{D}_\mu\Phi) - \lambda(\Phi^\dagger\Phi - v^2)^2\,, \tag{2.1}$$

with the field tensors

$$\begin{aligned} W^{\mu\nu} &= \partial^\mu W^\nu - \partial^\nu W^\mu + [W^\mu, W^\nu]\,, \\ Y^{\mu\nu} &= \partial^\mu Y^\nu - \partial^\nu Y^\mu\,, \end{aligned} \tag{2.2}$$

the covariant derivative

$$\mathcal{D}^\mu = \partial^\mu + W^\mu + Y^\mu\,, \tag{2.3}$$

and $\langle\cdot,\cdot\rangle$ the $\mathrm{su}(2)\oplus\mathrm{u}(1)$ inner product.

There is some freedom in the choice of non-degenerate inner products $\langle\cdot,\cdot\rangle$ on $\mathrm{su}(2)_I\oplus\mathrm{u}(1)_Y$, constrained to be invariant under the adjoint action of $\mathrm{SU}(2)_I\times\mathrm{U}(1)_Y$. We parameterise the possible inner products in the following way

$$\langle X, Y\rangle = -\frac{1}{g^2}(2\mathrm{tr}XY - \mathrm{tr}X\mathrm{tr}Y) - \frac{1}{g'^2}\mathrm{tr}X\mathrm{tr}Y\,, \tag{2.4}$$

with real, positive parameters g and g'. Choosing a basis $\{\frac{1}{2}i\sigma_a\}$ for su(2), with σ_a the Pauli spin matrices, and $\frac{1}{2}i\mathbf{1}_2$ for u(1), we see that the unit norm generators are

$$\left\|\frac{1}{2}ig\sigma_a\right\| = \left\|\frac{1}{2}ig'\mathbf{1}_2\right\| = 1\,. \tag{2.5}$$

With these generators the covariant derivative explicitly takes the form

$$\mathcal{D}^\mu = \partial^\mu + \frac{1}{2}ig\sigma_a W_a^\mu + \frac{1}{2}ig'Y^\mu\,, \tag{2.6}$$

yielding the familiar interpretation of g and g' as the isospin and hypercharge gauge coupling constants.

Symmetry breaking is seen through minimisation of the lagrangean (2.1), one solution of which is the vacuum solution

$$\Phi(x) = \Phi_0 = v\begin{pmatrix}0\\1\end{pmatrix}, \qquad W^\mu = Y^\mu = 0\,. \tag{2.7}$$

The other minima of (2.1) collectively give rise to the vacuum manifold of degenerate equivalent solutions

$$M = \mathrm{SU}(2)_I\times\mathrm{U}(1)_Y\cdot\Phi_0 = \{\Phi : \Phi^\dagger\Phi = v^2\}\,, \tag{2.8}$$

a three-sphere.

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Around the vacuum $\Phi(x) = \Phi_0$ the gauge field mass eigenstates are given by W_1^μ, W_2^μ, and

$$\begin{pmatrix} Z^\mu \\ Q^\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_w & -\sin\theta_w \\ \sin\theta_w & \cos\theta_w \end{pmatrix} \begin{pmatrix} W_3^\mu \\ Y^\mu \end{pmatrix}, \tag{2.9}$$

$$\begin{pmatrix} \alpha X_Z \\ e X_Q \end{pmatrix} = \begin{pmatrix} \cos\theta_w & -\sin\theta_w \\ \sin\theta_w & \cos\theta_w \end{pmatrix} \begin{pmatrix} \frac{1}{2} i g \sigma_3 \\ \frac{1}{2} i g' \mathbf{1}_2 \end{pmatrix}, \tag{2.10}$$

an orthogonal transformation of the fields, so that the covariant derivative becomes

$$\mathcal{D}^\mu = \partial^\mu + \sum_{i=1}^{2} \frac{1}{2} i g \sigma_i W_i^\mu + \alpha X_Z Z^\mu + e X_Q Q^\mu \,. \tag{2.11}$$

Interpreting Q^μ as the photon constrains $X_Q \Phi_0 = 0$, yielding $\tan\theta_w = g'/g$. This massless gauge field Q^μ is associated with the a residual $\mathrm{U}(1)_Q$ electromagnetic gauge symmetry

Explicitly the generators are

$$X_Z = \frac{1}{2} i \begin{pmatrix} \cos 2\theta_w & 0 \\ 0 & -1 \end{pmatrix}, \qquad X_Q = i \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.12}$$

with $\alpha = \sqrt{g^2 + g'^2}$, $e = gg'/\alpha$.

The massive gauge field generators $\{\frac{1}{2} i \sigma_1, \frac{1}{2} i \sigma_2, X_Z\}$ form an orthonormal basis for a vector space $\mathcal{M}$, such that

$$\mathrm{su}(2)_I \oplus \mathrm{u}(1)_Y = \mathrm{u}(1)_Q \oplus \mathcal{M} \,, \tag{2.13}$$

with the orthogonality determined by the inner product $\langle \cdot, \cdot \rangle$. This relation draws particular comparison to the isomorphism $M \cong \mathrm{SU}(2)_I \times \mathrm{U}(1)_Y / \mathrm{U}(1)_Q$, associating the space of massive generators with the tangent space to the vacuum manifold at Φ_0.

3. Vacuum geometry

For both the scalar and gauge sectors we now explicitly calculate their associated metrics. We also calculate the associated isometry and isotropy groups for each, and from these groups specify the corresponding geodesic structures.

Having obtained two inequivalent homogenous metrics on the vacuum manifold we then compare their structure. We relate their isometry and isotropy groups, and determine when curves are mutually geodesic with respect to both metrics.

3.1 Scalar sector

The structure associated with the scalar sector is a vector space of scalar field values $\mathbb{C}^2$ equipped with a real Euclidean inner product $\mathrm{Re}[\Psi^\dagger \Phi]$. Regarding the vacuum manifold as embedded within the vector space of scalar field values, a natural metric may be induced on the vacuum manifold by specifying its form on each tangent space to be that of the Euclidean inner product. Such a metric is isotropic and homogenous. Its geodesics are the great circles.

Explicitly, regard the tangent space to M at $\Phi \in M$ as an $\mathbb{R}^3$ subspace of $\mathbb{C}^2$

$$T_\Phi M = \{\Psi \in \mathbb{C}^2 : \mathrm{Re}[\Psi^\dagger \Phi] = 0\}, \tag{3.1}$$

with a corresponding metric induced from the real Euclidean inner product

$$g(T_1, T_2)_\Phi = \mathrm{Re}[T_1^\dagger T_2]\,, \qquad T_1, T_2 \in T_\Phi M\,. \tag{3.2}$$

This metric has an $\mathrm{SU}(2)_I \times \mathrm{SU}(2)_K$ group of isometries,

$$g(aT_1, aT_2)_{a\Phi} = g(T_1, T_2)_\Phi\,, \qquad a \in \mathrm{SU}(2)_I \times \mathrm{SU}(2)_K\,. \tag{3.3}$$

The $\mathrm{SU}(2)_I$ represents the usual left isospin $\mathrm{SU}(2)$ actions upon $\mathbb{C}^2$ with generators $\{\frac{1}{2}i\sigma_a\}$, whilst $\mathrm{SU}(2)_K$ acts upon $\mathbb{C}^2$ with the generators $\{-\frac{1}{2}\sigma_2 K, \frac{1}{2}i\sigma_2 K, -\frac{1}{2}i\mathbf{1}_2\}$, where K is the complex conjugation operator. This $\mathrm{SU}(2)_K$ contains the hypercharge $\mathrm{U}(1)_Y$, and some other additional symmetries of the scalar sector which are not symmetries of the full gauge theory.

The isotropy group of $\mathrm{SU}(2)_I \times \mathrm{SU}(2)_K$ upon M at the point Φ_0 is the subgroup $\mathrm{SU}(2)_{I-K}$ such that

$$\mathrm{SU}(2)_{I-K} \cdot \Phi_0 = \Phi_0\,, \tag{3.4}$$

which is generated by $\{\frac{1}{2}i\sigma_1 + \frac{1}{2}\sigma_2 K, \frac{1}{2}i\sigma_2 - \frac{1}{2}i\sigma_2 K, \frac{1}{2}i\sigma_3 + \frac{1}{2}i\mathbf{1}_2\}$. This gives the isomorphism

$$M \cong \frac{\mathrm{SU}(2)_I \times \mathrm{SU}(2)_K}{\mathrm{SU}(2)_{I-K}}\,. \tag{3.5}$$

One should be aware that $\mathrm{SU}(2) \times \mathrm{SU}(2)$ is the compact covering group of $\mathrm{SO}(4)$, just as $\mathrm{SU}(2)$ is the compact covering group of $\mathrm{SO}(3)$. Thus eq. (3.5) is an expression of the more familiar relation $S^3 \cong \mathrm{SO}(4)/\mathrm{SO}(3)$.

Given the above isotropy and isometry properties of the metric we can use the isomorphism (3.5) to calculate the geodesics upon the vacuum manifold with respect to the scalar sector metric $g(\cdot,\cdot)$. This geodesic structure follows from some results of differential geometry [1, chapter X]. Specifically, these results examine the geodesic structure on the coset space, but this may be simply carried back to $M \subset \mathbb{C}^2$ to give the results that we require. We summarise the full result in an appendix and give only the answer here.

We firstly need an inner product upon $\mathrm{su}(2)_I \oplus \mathrm{su}(2)_K$ which we shall define as

$$\{X_I + X_K, Y_I + Y_K\} = -2\mathrm{tr} X_I Y_I - 2\mathrm{tr} X_K Y_K\,, \tag{3.6}$$

with an obvious notation. This then induces

$$\mathrm{su}(2)_I \oplus \mathrm{su}(2)_K = \mathrm{su}(2)_{I-K} \oplus \mathcal{N}\,, \tag{3.7}$$

where $\mathcal{N}$ has an orthogonal basis $\{\frac{1}{2}i\sigma_1 - \frac{1}{2}\sigma_2 K, \frac{1}{2}i\sigma_2 + \frac{1}{2}i\sigma_2 K, \frac{1}{2}i\sigma_3 - \frac{1}{2}i\mathbf{1}_2\}$.

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The geodesic structure is then as follows:

The geodesics on M with respect to the metric $g(\cdot,\cdot)$, which pass through Φ_0 are:

$$\gamma_X = \{\exp(Xt)\Phi_0 : t \in \mathbb{R}\}, \tag{3.8}$$

with $X \in \mathcal{N}$.

The above geodesic structure is the main result of this section. Essentially we have derived the geodesic structure to consist of the great circles embedded in a three-sphere. This result is as expected, since we are merely embedding the three-sphere within a Euclidean space. However, it is the method which is of importance. The same approach may be adopted for the gauge sector, where the result is intuitively less clear. Also using the same formalism allows direct comparison between the metrical structures associated with the gauge and scalar sectors.

We conclude this section by exploring some consequences of the two eqs. (3.6) and (3.7). The space of geodesic generators $\mathcal{N}$ is related to the tangent space of M at Φ_0:

$$T_{\Phi_0}M = \mathcal{N}\cdot\Phi_0\,. \tag{3.9}$$

Considering a general $\Phi = a\Phi_0 \in M$ with $a \in \mathrm{SU}(2)_I \times \mathrm{SU}(2)_K$, there is a more general association

$$T_\Phi M = \mathrm{Ad}(a)\mathcal{N}\cdot\Phi\,. \tag{3.10}$$

For consistency the Euclidean inner product on $T_\Phi M$ is equivalent to the inner product on $\mathrm{Ad}(a)\mathcal{N}$ induced by $\{\cdot,\cdot\}$ of eq. (3.6)

$$g(X_1\Phi, X_2\Phi)_\Phi = \{X_1, X_2\}\,, \qquad X_i \in \mathrm{Ad}(a)\cdot\mathcal{N}\,. \tag{3.11}$$

This property is essential to the derivation of eq. (3.8), and motivates the choice made in eq. (3.6).

3.2 Gauge sector

The main structure associated with the gauge sector is the inner product $\langle\cdot,\cdot\rangle$ of eq. (2.4). It specifies several important related features of the gauge sector. First of all it defines the gauge kinetic term in eq. (2.1), introducing the gauge coupling constants as the relative scales. Secondly it stipulates the embedding of the massive gauge generators $\mathcal{M}$ in eq. (2.13) to be perpendicular to $\mathrm{u}(1)_Q$. Finally it renders the photon X_Q, Z-field X_Z and W-field generators mutually orthonormal.

We shall use this inner product to define the gauge-sector metric. The definition is achieved by associating the massive generators $\mathcal{M}$ with tangent spaces to the vacuum manifold in a manner completely analogous to that in the last section. Then the natural inner product $\langle\cdot,\cdot\rangle$ on the massive generators $\mathcal{M}$ induces a metric on the vacuum manifold. We find the corresponding isometry group of this metric to be

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the gauge group $\mathrm{SU}(2)_I \times \mathrm{U}(1)_Y$ and the isotropy group to be the residual symmetry $\mathrm{U}(1)_Q$. Thus the metric is homogenous, but anisotropic. Its geodesics are rather complicated in structure.

Explicitly, observe that the tangent space (3.1) may be expressed

$$T_{\Phi_0} M = \mathcal{M} \cdot \Phi_0 \,. \tag{3.12}$$

More generally, the corresponding tangent space at $\Phi = a\Phi_0 \in M$ is, for any $a \in \mathrm{SU}(2)_I \times \mathrm{U}(1)_Y$,

$$T_\Phi M = a T_{\Phi_0} M = \mathrm{Ad}(a) \mathcal{M} \cdot \Phi \,. \tag{3.13}$$

Transitivity over M guarantees a natural isomorphism between any tangent space and some $\mathrm{Ad}(a)\mathcal{M}$.

Using the isomorphism implied by eq. (3.13), the inner product $\langle \cdot, \cdot \rangle$ associates a corresponding metric on M

$$h(X_1\Phi, X_2\Phi)_\Phi = \langle X_1, X_2 \rangle \,, \qquad X_1, X_2 \in \mathrm{Ad}(a)\mathcal{M} \,. \tag{3.14}$$

The precise form is parameterised by the hypercharge and isospin coupling constants.

This metric has an $\mathrm{SU}(2)_I \times \mathrm{U}(1)_Y$ group of isometries

$$h(aT_1, aT_2)_{a\Phi} = h(T_1, T_2)_\Phi \,, \qquad a \in \mathrm{SU}(2)_I \times \mathrm{U}(1)_Y \,. \tag{3.15}$$

More precisely, by eq. (3.13) the action of $a \in \mathrm{SU}(2)_I \times \mathrm{U}(1)_Y$ upon $h(\cdot, \cdot)$ is

$$h(aT_1, aT_2)_{a\Phi} = \langle \mathrm{Ad}(a)X_1, \mathrm{Ad}(a)X_2 \rangle = \langle X_1, X_2 \rangle = h(T_1, T_2)_\Phi \,. \tag{3.16}$$

The above isometries represent the maximal subgroup of $\mathrm{SU}(2)_I \times \mathrm{SU}(2)_K$ leaving $\langle \cdot, \cdot \rangle$ invariant.

The isotropy group of this isometry group at the point Φ_0 in M is the subgroup $\mathrm{U}(1)_Q$ such that

$$\mathrm{U}(1)_Q \cdot \Phi_0 = \Phi_0 \,, \tag{3.17}$$

giving the isomorphism

$$M \cong \frac{\mathrm{SU}(2)_I \times \mathrm{U}(1)_Y}{\mathrm{U}(1)_Q} \,. \tag{3.18}$$

Thus we recover the familiar relation for the vacuum manifold, but now explicitly associated with the gauge sector metrical structure.

As for the scalar sector the importance of isomorphism (3.18) is that we may use the isotropy and isometry properties of the metric to calculate the geodesics upon the vacuum manifold with respect to the gauge sector metric. Again we give only answer and refer to the full result summarised in the appendix.

The structure is:

The geodesics on M with respect to the metric $h(\cdot,\cdot)$, passing through Φ_0 are:

$$\gamma_X = \{\exp(Xt)\Phi_0 \,:\, t \in \mathbb{R}\}\,, \tag{3.19}$$

with $X \in \mathcal{M}$.

A short calculation yields the geodesics through Φ_0 associated with the following generators in $\mathcal{M}$

$$X = \frac{1}{2} i c_1 g \sigma_1 + \frac{1}{2} i c_2 g \sigma_2 + d X_Z\,, \qquad c_1^2 + c_2^2 + d^2 = 1 \tag{3.20}$$

to be

$$\gamma_X(t) = v e^{-id\tan^2\theta_w t/2} \begin{pmatrix} c \sin t/2 \\ \cos t/2 - id \sin t/2 \end{pmatrix}, \tag{3.21}$$

where $c = c_2 + ic_1$. This structure is rather complicated, for instance the closed geodesics from a discrete set such that

$$\begin{aligned} d\tan^2\theta_w &\in \mathbb{Q}\,, \qquad c \neq 0\,, \\ d &\in \mathbb{Q}\,, \qquad c = 0\,. \end{aligned} \tag{3.22}$$

Geodesics through other points $\Phi = a\Phi_0 \in M$ may be simply obtained by acting correspondingly on eq. (3.21).

There are a couple of points to observe about the geodesic structure. Firstly, their exists a totally geodesic two sphere through Φ_0 defined by those geodesics satisfying $d = 0$. Secondly, in the direction perpendicular to the tangent space of this sphere there exists a closed geodesic satisfying $c = 0$. Because of homogeneity this is also true for any point on the vacuum manifold.

The above metric is homogenous but anisotropic. Its anisotropy is parametrised by the weak mixing angle θ_w, becoming isotropic when θ_w vanishes. Thus one may interpret the gauge metric $h(\cdot,\cdot)$ as a continuous and homogenous deformation of the isotropic Euclidean scalar metric $g(\cdot,\cdot)$. At each point this deformation leaves a two-sphere and a circle unaffected, but deforms relative to the two. In some sense this structure is analogous to *squashing* the three-sphere; having the geometry of an ellipsoid, but with the deformation homogeneous so as to respect the homogeneity of (3.18).

We conclude this part of the discussion by observing that the above structure is rather special to the three-sphere. It is related to the Hopf fibration. In the Hopf fibration picture the metric on the S^1 fibres has a different length scale to that on the S^2 base space. One should note that the only other spheres to have a similar structure are the seven-sphere and the fifteen-sphere.

3.3 Scalar-gauge geometry

In summary, we found two inequivalent metrics on the electroweak vacuum manifold associated with the scalar and gauge sectors. We shall now determine how the structure of these metrics relate to each other. Comparing the respective symmetry groups determines those symmetries which are shared. These shared symmetries define two submanifolds whose geodesics are mutually geodesic with respect to the two metrics. These correspond to the two-sphere and the circle mentioned in the discussion of the gauge sector metric.

The scalar and gauge metrics, $g(\cdot,\cdot)$ and $h(\cdot,\cdot)$, have the following isometry group decompositions with respect to their isotropy groups

$$\begin{aligned} \mathrm{su}(2)_I \oplus \mathrm{su}(2)_K &= \mathrm{su}(2)_{I-K} \oplus \mathcal{N}\,, \\ \mathrm{su}(2)_I \oplus \mathrm{u}(1)_Y &= \mathrm{u}(1)_Q \oplus \mathcal{M}\,, \end{aligned} \tag{3.23}$$

where the group structure is related by

$$\begin{aligned} \mathrm{u}(1)_Q &\subset \mathrm{su}(2)_{I-K}\,, \\ \mathrm{u}(1)_Y &\subset \mathrm{su}(2)_K\,, \\ \mathcal{M} &\subset \mathrm{su}(2)_I \oplus \mathrm{su}(2)_K\,. \end{aligned} \tag{3.24}$$

Also, the tangent space to M at Φ_0 is related to $\mathcal{M}$ and $\mathcal{N}$ by

$$T_{\Phi_0}M = \mathcal{M}\cdot\Phi_0 = \mathcal{N}\cdot\Phi_0\,. \tag{3.25}$$

It is important to understand how the metrics $g(\cdot,\cdot)$ and $h(\cdot,\cdot)$ relate to each another. By bilinearity of the metrics, at the point $\Phi_0 \in M$,

$$g(T_1,T_2)_{\Phi_0} = h(AT_1,AT_2)_{\Phi_0}\,, \qquad A \in \mathrm{GL}(T_{\Phi_0}M)\,, \tag{3.26}$$

relating the metrics by a linear map of the tangent space. The eigenspaces of A can be found by explicitly calculating A, and diagonalising it, This yields

$$T_{\Phi_0}M = T^Z_{\Phi_0}M \oplus T^W_{\Phi_0}M\,, \tag{3.27}$$

with

$$\begin{aligned} T^Z_{\Phi_0}M &= \mathcal{M}_Z\cdot\Phi_0\,, \qquad & \mathcal{M}_Z &= \mathbb{R}\cdot X_Z\,, \\ T^W_{\Phi_0}M &= \mathcal{M}_W\cdot\Phi_0\,, \qquad & \mathcal{M}_W &= \mathbb{R}\cdot\frac{1}{2}i\sigma_1 \oplus \mathbb{R}\cdot\frac{1}{2}i\sigma_2\,. \end{aligned} \tag{3.28}$$

Then the metrics are related such that

$$g(X_Z\Phi_0 + X_W\Phi_0, Y_Z\Phi_0 + Y_W\Phi_0)_{\Phi_0} = \lambda_Z h(X_Z\Phi_0, Y_Z\Phi_0)_{\Phi_0} + \lambda_W h(X_W\Phi_0, Y_W\Phi_0)_{\Phi_0}\,, \tag{3.29}$$

with an obvious notation, and $\lambda_Z = \alpha^2 v^2 / \cos 2\theta_w$, $\lambda_W = g^2 v^2$. This may be easily generalised to all $\Phi \in M$ by considering the action of $\mathrm{SU}(2)_I \times \mathrm{U}(1)_Y$ on eq. (3.29).

Decomposition (3.27) also describes the geodesic structure of M with respect to $g(\cdot,\cdot)$ and $h(\cdot,\cdot)$ in a rather nice way. Applying eqs. (3.8) and (3.19), the submanifolds

$$\begin{aligned} M^Z &= \exp(\mathcal{M}_Z)\Phi_0\,, \qquad T_{\Phi_0} M^Z = T^Z_{\Phi_0} M \\ M^W &= \exp(\mathcal{M}_W)\Phi_0\,, \qquad T_{\Phi_0} M^W = T^W_{\Phi_0} M \end{aligned} \tag{3.30}$$

are the *only* totally geodesic submanifolds of M with respect to *both* metrics $g(\cdot,\cdot)$ and $h(\cdot,\cdot)$. No other submanifold of M has this property.

One should note that the $\mathrm{Ad}(\mathrm{U}(1)_Q)$-irreducible subspaces of $\mathcal{M}$ with respect to the inner product $\langle\cdot,\cdot\rangle$ are

$$\mathcal{M} = \mathcal{M}_Z \oplus \mathcal{M}_W\,, \tag{3.31}$$

as found in the decomposition above. These also relate to the mass eigenstates of the massive gauge fields. This property is in fact very general [2].

4. Physical implications

In summary of the previous section we found two homogenous metrics on the vacuum manifold associated with the scalar and gauge sectors of Weinberg-Salam theory. The scalar sector induces an isotropic metric, whilst the gauge sector induces an anisotropic metric. There is a unique totally geodesic two sphere with respect to both metrics. Geodesic curves with respect to both metrics consist of the geodesics in this two sphere, and one other circular path whose tangent vector is orthogonal to the tangent plane of this two sphere.

Given this structure one might inquire as to how this relates to the spectrum of non-perturbative solitonic-type solutions present in Weinberg-Salam theory. It transpires that the electroweak strings correspond to the mutually geodesic paths, whilst the sphaleron corresponds to the mutually geodesic two-sphere. This approach has the added bonus of interpreting the scattering of electroweak strings in terms of the holonomy of their respective geodesic. Also the dynamical stability of the Z-string, as the weak mixing angle approaches $\pi/2$, is seen to correspond to extreme anisotropy of the gauge metric.

4.1 Electroweak strings

Electroweak strings correspond to Nielsen-Olesen vortices embedded in Weinberg-Salam theory [3]. As such their boundary conditions define circular paths on the vacuum manifold. Thus one might expect that their spectrum and properties should correspond to the geometry of the vacuum manifold. This is what we find. Their boundary conditions correspond to the paths that are mutually geodesic with respect to *both* metrics.

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Formally, an electroweak string is defined by the embedding

$$\begin{array}{ccc} \mathrm{SU}(2)_I \times \mathrm{U}(1)_Y & \rightarrow & \mathrm{U}(1)_Q \\ \cup & & \cup \\ \mathrm{U}(1) & \rightarrow & 1\,, \end{array} \tag{4.1}$$

with the general Ansatz

$$\begin{aligned} \Phi(r,\theta) &= f_{\mathrm{NO}}(r) e^{X\theta}\Phi_0\,, \\ \underline{A}(r,\theta) &= \frac{g_{\mathrm{NO}}(r)}{r} X \underline{\hat{\theta}}\,, \end{aligned} \tag{4.2}$$

where $X \in \mathrm{su}(2)_I \oplus \mathrm{u}(1)_Y$ is the vortex generator. One may consider only $X \in \mathcal{M}$, since these minimise the magnetic energy [4]. Thus one considers only Ansätze with boundary conditions geodesic with respect to the scalar metric $h(\cdot,\cdot)$.

The above vortex Ansatz is a solution provided that [5]

i) The scalar field must be single-valued. Hence the boundary conditions describe a closed geodesic with $e^{2\pi X}\Phi_0 = \Phi_0$.

ii) The Ansatz is a solution to the equations of motion; then fields in the vortex do not induce currents perpendicular (in Lie algebra space) to it [6]. This may be equivalently phrased as [4]: X is a vortex generator if $\mathrm{Re}[(X\Phi_0)^\dagger X^\perp \Phi_0] = 0$ for all $X^\perp$ such that $\langle X^\perp, X\rangle = 0$.

Condition ii) can be conveniently restated in terms of the corresponding metrics:

X is a vortex generator if the associated tangent vector $T = X\Phi_0$ satisfies $g(T, T^\perp) = 0$ for all $T^\perp$ such that $h(T, T^\perp) = 0$.

Referring to the discussion around eq. (3.27), we see that T must lie in one of the eigenspaces of the linear map relating the two metrics. Namely X is an element of either $\mathcal{M}_Z$ or $\mathcal{M}_W$, the two $\mathrm{Ad}(\mathrm{U}(1)_Q$-irreducible subspaces of $\mathcal{M}$ in the decomposition

$$\mathcal{M} = \mathcal{M}_Z \oplus \mathcal{M}_W\,. \tag{4.3}$$

It is interesting that the geodesics defined from $\mathcal{M}_Z$ and $\mathcal{M}_W$ are the only geodesics which are *simultaneously geodesic with respect to both metrics.* From a mathematical point of view this is because these geodesics define submanifolds of the vacuum manifold with coincident scalar and gauge metrics (to an overall factor). From a physical point of view, this may be interpreted as a minimisation of both the scalar and gauge sectors of the action integral.

From the above it is a fairly trivial exercise to work out the forms of the vortices, obtaining results in agreement with refs. [3, 5].

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4.2 Semi-local vortices

When the weak mixing angle becomes $\pi/2$ the isospin gauge symmetry $\mathrm{SU}(2)_I$ becomes a global symmetry whilst the hypercharge symmetry $\mathrm{U}(1)_Y$ remains local. Such a model is interpreted as a complex doublet scalar field with a gauged phase. For suitable scalar potentials it admits dynamically stable semi-local vortex solutions [7], interpreted as the corresponding limit of a Z-string. By continuity the Z-string is dynamically stable in a region close to weak mixing angle $\pi/2$ [3]. We wish to point out that this is related to the gauge metric becoming *extremely* anisotropic.

As the weak mixing angle tends to $\pi/2$ the isospin coupling g tends to zero. For $g = 0$, the inner product $\langle\cdot,\cdot\rangle$ becomes ill defined. It is well defined only on the subalgebra $\mathrm{u}(1)_Y$, where it takes the form

$$\langle X, Y\rangle_{\pi/2} = -\frac{1}{g'^2}\mathrm{tr}X\,\mathrm{tr}Y\,. \tag{4.4}$$

By eqs. (3.14), (3.30), the gauge metric is only defined upon the one-dimensional submanifold $M_Z \subset M$, where

$$h_{\pi/2}(X_1\Phi, X_2\Phi)_\Phi = \langle X_1, X_2\rangle_{\pi/2}\,, \qquad X_1, X_2 \in \mathcal{M}_Z\,. \tag{4.5}$$

This is interpreted as the limit of *extreme* anisotropy; this anisotropy picking out the submanifold $M_Z \subset M$ upon which the metric is well defined.

Physically, the submanifold M_Z represents those points that may be reached by a gauge transformation from Φ_0. Other points within the vacuum manifold may only be reached by a global transformation. This property is related to the stability of the vortex. To decay, the vortex solution must deform out of M_Z. Such a process costs large gradient energies that may not be compensated for by a gauge field [7].

4.3 Combination electroweak vortices

When the weak mixing angle vanishes the hypercharge gauge symmetry $\mathrm{U}(1)_Y$ becomes a global symmetry whilst the isospin symmetry $\mathrm{SU}(2)_I$ remains local. Such a model is interpreted as a gauged complex doublet scalar field with a global phase. Then Weinberg-Salam theory represents the symmetry breaking $\mathrm{SU}(2)_I \to \mathbf{1}$. Correspondingly there is a two-parameter family of embedded vortices representing Z-strings, W-strings, and combination W-Z strings. We wish to point out here that this is related to the gauge metric becoming isotropic, coinciding with the scalar metric and thus all geodesics of the scalar sector define electroweak strings.

When the weak mixing angle vanishes, so that $g' = 0$, the inner product $\langle\cdot,\cdot\rangle$ becomes well defined only on the subalgebra $\mathrm{su}(2)_I$,

$$\langle X, Y\rangle_0 = -\frac{1}{g^2}(2\mathrm{tr}XY - \mathrm{tr}X\,\mathrm{tr}Y)\,. \tag{4.6}$$

Analogous to eq. (3.14), the metric is then defined from $\langle\cdot,\cdot\rangle_0$ to be

$$h_0(X_1\Phi, X_2\Phi)_\Phi = \langle X_1, X_2\rangle_0\,, \qquad X_1, X_2 \in \mathrm{su}(2)_I\,. \tag{4.7}$$

Then the scalar and gauge metrics coincide upto a factor of the isospin coupling constant g

$$g(T_1, T_2) = \frac{1}{g} h_0(T_1, T_2)\,. \tag{4.8}$$

Thus the isometry group of the gauge metric increases to $\mathrm{SU}(2)_I \times \mathrm{SU}(2)_K$, with this $\mathrm{SU}(2)_K$ now representing a set of global SU(2) symmetries of the gauge theory only apparent at vanishing weak mixing angle.

In terms of the electroweak vortex spectrum, condition ii) is satisfied for all vortex generators $X \in \mathrm{su}(2)_I$, since with respect to the two metrics all vortices are trivially coincident. Hence the spectrum of vortex solutions becomes a continuous family defined by elements $X \in \mathrm{su}(2)_I$.

It is an interesting question to enquire what happens to these solutions as the weak mixing angle moves off zero. By continuity one might expect some sort of perturbed solution to exist. However it is difficult to see what its boundary conditions would be since necessarily they can only be a geodesic of either the gauge metric or the scalar metric, not both.

4.4 Non-abelian Aharanov-Bohm scattering

It is known that the Aharov-Bohm scattering of particles off a vortex, or one vortex off another, is controlled by the holonomy of a vortex's boundary conditions [8]. In this context the term holonomy was used to indicate non-trivial parallel transport of either a vortex or a charged particle in a circuit around a vortex. We show here that this holonomy refers precisely to the holonomy with respect to the gauge metric.

Associated with the magnetic flux of a vortex is the Wilson line integral

$$\mathrm{U}(\theta) = P\exp\left(\int_0^\theta \mathbf{A}\cdot d\mathbf{l}\right) \subset G\,, \tag{4.9}$$

at infinite radius. The function $\mathrm{U}(2\pi)$ dictates the parallel transport of matter fields around a vortex, such that a fermion doublet Ψ is transported to $\mathrm{U}(2\pi)\Psi$. Diagonalisation of U then associates components Ψ_i with phase shifts ξ_i. Non-trivial fermionic components Ψ_i interact with the vortex by an Aharanov-Bohm cross section

$$\frac{d\sigma}{d\theta} = \frac{1}{2\pi k}\frac{\sin^2\left(\frac{\xi_i}{2}\right)}{\sin^2\left(\frac{\theta}{2}\right)}\,, \tag{4.10}$$

whilst trivial components $e^{2\pi\xi_k} = 1$ interact by an Everett cross section [9]. The above holds for parity symmetric theories only. When charges for the left and right fermion fields differ one must also include the effects of induced fermionic zero modes.

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Substitution of the boundary conditions for the Z and W-string, determined in sec. (4.1), gives the Wilson line integral for electroweak strings. For the Z-string

$$U_Z(\theta) = \exp(-2X_Z\theta)\,, \tag{4.11}$$

whilst for the W-string

$$U_W(\theta) = \exp(i(\sigma_1 \sin\eta + \sigma_2 \cos\eta)\theta)\,. \tag{4.12}$$

Vortex boundary conditions restrict $\mathrm{U}(2\pi) \in \mathrm{U}(1)_Q$, where explicitly $U_Z(2\pi) = \exp(-4\pi\cos^2\theta_w X_Q)$ and $U_W(2\pi) = \mathbf{1}$. For the scattering of a fermion doublet Ψ off a Z-string this yields phase shifts $\xi_1 = -2\pi\cos^2\theta_w$ and $\xi_2 = 2\pi$. Thus upper components interact with Z-strings by an Aharanov-Bohm cross section, whilst lower components interact by an Everett cross section. Fermions interact with W-strings only by an Everett cross section.

In [8] the Wilson line integral is related to holonomy, referring to non-trivial parallel transport around the vortex. Referring to our appendix, in particular the result

the parallel transport of $\mathbf{u} \in T_{\mathbf{v}}M$ *along* $\tilde{\gamma}_X(t)$ *is* $\mathbf{u}' = D(\exp(Xt))\mathbf{u}$.

we see that this holonomy refers precisely to the *parallel transport with respect to the gauge sector metric around its closed geodesics*. For vortices relevant paths are geodesic with respect to both metrics.

One should be aware that parity violation in the standard model means that one must take into account the fermionic zero modes. This is done in ref. [10].

4.5 The sphaleron

Finally, we point out that the existence of the sphaleron solution in Weinberg-Salam theory is related to the presence an embedded two sphere in the vacuum manifold that is totally geodesic with respect to both the gauge and scalar metrics. The dipole moment of a sphaleron is also related to this embedding.

For $\theta_w = 0$ the Ansatz

$$\Phi(\mathbf{r}) = f_{\mathrm{sph}}(r)\exp\left(\frac{i\pi}{2}\hat{r}_a\sigma_a\right)\Phi_0\,, \tag{4.13}$$

$$A_a(\mathbf{r}) = g_{\mathrm{sph}}(r)\frac{i}{2}\epsilon_{abc}\hat{r}_b\sigma_c\,, \tag{4.14}$$

constitutes a solution to Weinberg-Salam theory for suitable profile functions f_{sph}, g_{sph}. It is unstable because the boundary conditions define a topologically trivial map $S^2 \to \mathrm{SU}(2)$. By continuity this solution is expected to persist to non-zero θ_w, with perturbations being produced in the fields. Such a solution is referred to as the Sphaleron [11].

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The scalar field asymptotically maps onto the submanifold $M^W \subset M$. Thus, as with electroweak strings, the boundary conditions define a totally geodesic submanifold of the vacuum manifold with respect to *both* the scalar and gauge metrics. One should note that M^W is the only two-dimensional submanifold of the vacuum manifold defined so.

To prevent an electric monopole component at non-zero weak mixing angle, the gauge field (4.14) deforms such that $\frac{i}{2}\sigma_3 \to X_Z$. This implies a preferred axis in the $\underline{\hat{r}}^3$-direction, with the configuration rotationally symmetric about it. Inducing, to lowest order, dipolar perturbations in the electromagnetic field

$$\delta Q^\mu = \frac{\underline{\mu} \wedge \underline{r}}{4\pi r^3} X_Q\,, \tag{4.15}$$

with $\underline{\mu}$ parallel to $\underline{\hat{r}}^3$, as found by substitution into the field equations [11].

5. Discussion

In this final section we briefly discuss some extensions to this work and make some comments that may warrant further note.

The general case The group theory in this paper can be extended to the general symmetry breaking $G \to H$ in a fairly straightforward manner. Hence, in general, one may expect two metrics on the vacuum manifold relating to the scalar and gauge sector. As in Weinberg-Salam theory, embedded vortices will be geodesic with with respect to both metrics, and the Aharonov-Bohm scattering will relate to the holonomy of the gauge sector metric.

Simplicity of electroweak theory Crucial to establishing the scalar and gauge metrics on the vacuum manifold was establishing the isomorphisms with the coset spaces $\mathrm{SU}(2)_K \times \mathrm{SU}(2)_I / \mathrm{SU}(2)_{I-K}$ and $\mathrm{SU}(2)_I \times \mathrm{U}(1)_Y / \mathrm{U}(1)_Q$. The first of which is a symmetric space, and the second is a non-symmetric homogenous space, interpreted as deformed from the first. This structure is in fact quite special, and electroweak theory constitutes the smallest dimensional example of this.

Energetics and curvature From the metrical structure one has a corresponding curvature of the vacuum manifold. It seems sensible that the energy of embedded vortices should be associated with the sum of the curvatures of the scalar and gauge metrics on the submanifold of the vacuum manifold associated with the scalar boundary conditions of the vortex. Coefficients of this sum should be related to those of the Landau potential, and the value of this sum should be related to the stability of the vortex.

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Insensitivity to the form of the theory In relating the solitonic spectrum and properties to the metrical structure of the theory one moves away from the specific details of the lagrangean. Thus the metrical approach relates more to the general symmetry features of the theory rather than the specific model of symmetry breaking.

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A. Homogenous spaces and geometry

We provide here a quick summary of the results of [1, chapter X] that are relevant to this work.

Consider a manifold G/H, where $H \subset G$ are compact Lie groups. Then an important and relevant decomposition is the reductive decomposition of the Lie algebras

$$\mathcal{G} = \mathcal{H} \oplus \mathcal{M}\,, \tag{A.1}$$

which satisfies

$$\begin{aligned} \mathrm{Ad}(H)\mathcal{H} &\subseteq \mathcal{H}\,, \\ \mathrm{Ad}(H)\mathcal{M} &\subseteq \mathcal{M}\,. \end{aligned} \tag{A.2}$$

Here $\mathcal{M}$ may be associated with the tangent space to M at the trivial coset. One should note that strictly speaking an inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{G}$ is required to define the reductive decomposition.

Given such a decomposition we can associate a G-invariant connection on G/H having the following properties:

i) All geodesics in M emanating from the trivial coset are of the form

$$\gamma_X(t) = \exp(Xt)H\,, \tag{A.3}$$

with $X \in \mathcal{M}$.

ii) Considering $\exp(Xt) \subset G$, the parallel transport of a tangent vector $Y \in \mathcal{M}$ along the curve $\exp(Xt)H$ is

$$Y' = \mathrm{Ad}(\exp(Xt))Y\,. \tag{A.4}$$

iii) The corresponding G-invariant metric on G/H is associated with the inner products on $\mathcal{G}$ such that at the trivial coset

$$g(X,Y)_H = \langle X,Y\rangle\,, \tag{A.5}$$

identifying tangent vectors with elements of $\mathcal{M}$.

By applying the above result one may examine the case with a vector space V such that G acts on it by the D-representation. Then $H_{\mathbf{v}}$ is the isotropy subgroup at $\mathbf{v}\in V$, and we associate the manifold

$$M = D(G)\mathbf{v} \cong \frac{G}{H}\,. \tag{A.6}$$

Given an inner product $\langle\cdot,\cdot\rangle$ on $\mathcal{G}$ we associate a G-invariant connection on M from the decomposition $\mathcal{G} = \mathcal{H}_{\mathbf{v}} \oplus \mathcal{M}$. Denoting the tangent space at $\mathbf{v}$ to M by $T_{\mathbf{v}}M = \mathcal{M}\mathbf{v}$, the following properties are apparent

i) All geodesics emanating from $\mathbf{v}$ are of the form

$$\tilde{\gamma}_X(t) = D(\exp(Xt))\mathbf{v}\,. \tag{A.7}$$

ii) The parallel transport of $\mathbf{u}\in T_{\mathbf{v}}M$ along $\tilde{\gamma}_X(t)$ is

$$\mathbf{u}' = D(\exp(Xt))\mathbf{u}\,. \tag{A.8}$$

iii) The corresponding G-invariant metric on M takes the value at $\mathbf{v}$

$$g(X\mathbf{v},Y\mathbf{v})_{\mathbf{v}} = \langle X,Y\rangle\,. \tag{A.9}$$

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